Differential Geometry IV: General Relativity

2.1 On Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$, let $(x^0, \dots, x^n)$ be the standard Cartesian coordinate system. Compute the induced metric on the submanifolds

$$S^{(n,1)}_{-1} = \Big\{ -(x^0)^2 + \sum_{i=1}^{n} (x^i)^2 = -1 \Big\}$$

and

$$S^{(n,1)}_{+1} = \Big\{ -(x^0)^2 + \sum_{i=1}^{n} (x^i)^2 = +1 \Big\}$$

(the latter is known as *de-Sitter* spacetime).

Solution. In order to compute the induced metric on those manifolds, we first have to find a convenient parametrization of them. For $S^{(n,1)}_{-1}$, it is convenient to express it as the graph of a function over the $\{x^0 = 0\}$ hyperplane; that is to say, we define $\Psi : \mathbb{R}^n \to S^{(n,1)}_{-1}$ by the relation:

$$\Psi(y^1, \dots, y^n) = (x^0, \dots, x^n) \quad \text{with} \quad x^\alpha = \begin{cases} \sqrt{1 + \|y\|^2}, & \text{if } \alpha = 0, \\ y^\alpha, & \text{if } \alpha \geqslant 1, \end{cases}$$

where

$$\|y\|^2 \doteq \sum_{i=1}^{n} (y^i)^2.$$

(Ψ defined as above parametrizes only one of the two components of $S^{(n,1)}_{-1}$; the other one is parametrized using $x^0 = -\sqrt{1 + \|y\|^2}$ instead of $x^0 = +\sqrt{1 + \|y\|^2}$, but the resulting expression for the induced metric is the same). Then, in the coordinate chart Ψ^{-1} on $S^{(n,1)}_{-1}$, the induced metric $g = \Psi^* \eta$ takes the form:

$$\begin{aligned} g = \Psi^*\eta &= -\Psi^*(dx^0)^2 + \sum_{i=1}^{n} \Psi^*(dx^i)^2 \\ &= -\Big(d\big(\sqrt{1 + \|y\|^2}\big)\Big)^2 + \sum_{i=1}^{n} (dy^i)^2 \\ &= -\frac{\big(\sum_{i=1}^{n} y^i dy^i\big)^2}{1 + \|y\|^2} + \sum_{i=1}^{n} (dy^i)^2 \\ &= \Big(\delta_{ij} - \frac{y^i y^i}{1 + \|y\|^2}\Big) dy^i dy^j. \end{aligned}$$

Notice that the above metric is Riemannian, i.e. positive definite, since the matrix $\delta_{ij} - \frac{y^i y^i}{1+\|y\|^2}$ is positive definite (here, we used the fact that for any vector $v \in \mathbb{R}^n$, the matrix $\mathbb{I} - v \otimes v$ has eigenvalues $(1 - \|v\|^2, 1, \dots, 1)$). This is, of course, to be expected, since $S^{(n,1)}_{-1}$ is a *spacelike* hypersurface of $(\mathbb{R}^{n+1}, \eta)$. In polar coordinates (r, ω) on $\mathbb{R}^n \setminus 0 \simeq (0, +\infty) \times \mathbb{S}^{n-1}$, g takes the form

$$g = \frac{1}{1 + r^2} dr^2 + r^2 g_{\mathbb{S}^{n-1}}(\omega).$$

The metric g is in fact isometric to the *hyperbolic* metric on $\mathbb{R}^n$.

Since $S_+^{(n,1)}$ has the topology of a cylinder, we will use for it a parametrization by $\mathbb{R}\times\mathbb{S}^{n-1}$. To this end, let us switch to polar coordinates (x^0, r, ω) on $\mathbb{R}^{n+1}$ (so that $r=\sqrt{\sum_{i=1}^n (x^i)^2}\in[0,+\infty)$ and $\omega\in\mathbb{S}^{n-1}$ with $\omega^i=\frac{x^i}{r}$, $i\geqslant 1$). In these coordinates, the Minkowski metric η takes the form:

$$\eta = -(dx^0)^2 + dr^2 + r^2 g_{\mathbb{S}^{n-1}}(\omega),$$

while $S_+^{(n,1)}=\Big\{(x^0,r,\omega):\,-(x^0)^2+r^2=1\Big\}$. Let us consider the parametrization $\Phi:\mathbb{R}\times\mathbb{S}^{n-1}\to S_+^{(n,1)}$ given by

$$\Phi(t,\theta)=(x^0,r,\theta)\quad\text{with}\quad x^0=t,\, r=\sqrt{1+t^2},\, \theta=\omega.$$

Then, the induced metric g_{dS} on $S_+^{(n,1)}$ (the so-called *de-Sitter metric*) takes the following form

$$\begin{aligned}
g_{dS}=\Phi_*\eta &= -\Phi_*(dx^0)^2+\Phi_*dr^2+r^2\circ\Phi\cdot\Phi_*g_{\mathbb{S}^{n-1}}(\omega)\\
&= -dt^2+\Big(d\big(\sqrt{1+t^2}\big)\Big)^2+(1+t^2)g_{\mathbb{S}^{n-1}}(\theta)\\
&= -\frac{1}{1+t^2}dt^2+(1+t^2)g_{\mathbb{S}^{n-1}}(\theta).
\end{aligned}$$

2.2 On Minkowski spacetime $(\mathbb{R}^{n+1},\eta)$, let $p,q\in\mathbb{R}^{n+1}$ be two points such that $q\in I^+(p)$. Let also $\gamma_0:[0,1]\to\mathbb{R}^{n+1}$ be the straight line segment connecting p to q (i.e. $\gamma_0(0)=p$, $\gamma_0(1)=q$ and $\ddot\gamma_0=0$) and $\gamma:[0,1]\to\mathbb{R}^{n+1}$ be any other *causal* curve such that $\gamma(0)=p$ and $\gamma(1)=q$. Show that the corresponding lengths of the curves satisfy

$$\ell(\gamma_0)\geqslant\ell(\gamma).$$

This is a manifestation of the *twin paradox* in special relativity.
Hint: Approximate γ by a polygonal causal curve and, using the inverse triangle inequality for causal vectors, show that the line segment connecting p and q has greater or equal length to a broken line segment connecting the same points.

Solution. Let us first assume that the curve γ is a *polygonal*, future directed causal curve joining p and q, that is to say, there exist points $\{p_k\}_{k=0}^N\in\mathbb{R}^{n+1}$ such that

1. $p_0=p$, $p_N=q$,
2. The curve γ is the union of the line segments $\overrightarrow{p_{k-1}p_k}$ connecting p_{k-1} to p_k; explicitly, this means that there exists some partition $\{t_k\}_{k=0}^N$ of $[0,1]$ such that $t_0=0$, $t_N=1$ and

$$\gamma(s)=\frac{t_{k+1}-s}{t_{k+1}-t_k}p_k+\frac{s-t_k}{t_{k+1}-t_k}p_{k+1}\quad\text{if } s\in[t_k,t_{k+1}].$$

3. For all $k=0,\dots,N-1$, the vectors $\overrightarrow{p_{k-1}p_k}$ are *future directed* and causal.

Recall that the inverse triangle inequality states that, in any Lorentzian inner product space (V, m), if v, w are two causal vectors belonging to the same component of the timecone, then

$$\|v + w\| \geqslant \|v\| + \|w\|.$$

In our case, noting that

$$\overrightarrow{pq} = \sum_{k=0}^{N-1} \overrightarrow{p_k p}_{k+1},$$

we have (thinking of those vectors as belonging to $T_p\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$ by translating them to have a starting point at p):

$$\|\overrightarrow{pq}\|_{\eta_p} \geqslant \sum_{k=0}^{N-1} \|\overrightarrow{p_k p}_{k+1}\|_{\eta_p}.$$

Noting that, in this case

$$\ell(\gamma_0) = \|\overrightarrow{pq}\|_{\eta_p} \quad \text{and} \quad \ell(\gamma) = \sum_{k=0}^{N-1} \|\overrightarrow{p_k p}_{k+1}\|_{\eta_p},$$

we therefore have

$$\ell(\gamma_0) \geqslant \ell(\gamma).$$

Assume, now, that γ is a C^1 causal curve; in that case, since $\dot{\gamma}$ is continuous, it has to belong everyewhere to the same connected component of the timecone, therefore it has to be future directed (since $q \in I^+(p)$). We will show that, for any $\epsilon > 0$ sufficiently small, there exists a point $q_\epsilon \in I^+(p)$ and a *polygonal* future directed, causal curve $\gamma_\epsilon : [0,1] \to \mathbb{R}^{n+1}$ with $\gamma_\epsilon(0) = p$, $\gamma_\epsilon(1) = q_\epsilon$ and such that:

- $q_\epsilon \to q$ as $\epsilon \to 0$,
- $\big|\ell(\gamma_\epsilon) - \ell(\gamma)\big| \to 0$ as $\epsilon \to 0$.

Assume for a moment that such a point q_ϵ and curve γ_ϵ indeed exists for all sufficiently small $\epsilon > 0$. Then, since γ_ϵ is a polygonal curve, by our previous argument we have

$$\|\overrightarrow{pq}_\epsilon\|_{\eta_p} \geqslant \ell(\gamma_\epsilon).$$

As $\epsilon \to 0$, we have (by our assumptions on q_ϵ, γ_ϵ) that $q_\epsilon \to q$ and $\ell(\gamma_\epsilon) \to \ell(\gamma)$. Therefore, since $\|\overrightarrow{pq}\|_{\eta_p} = \ell(\gamma_0)$, taking the limit $\epsilon \to 0$ in the above inequality results in the desired statement:

$$\ell(\gamma_0) \geqslant \ell(\gamma).$$

It, therefore, remains to construct the approximating point q_ϵ and polygonal curve γ_ϵ satisfying Conditions 1 and 2 above. To this end, let us first note that any piecewise C^1 curve $\zeta : [0,1] \to \mathbb{R}^{n+1}$

can be obtained from its derivative $\dot{\zeta} : [0,1] \to \mathbb{R}^{n+1}$ (here we view $\dot{\zeta}(t)$ as a vector with base point translated to 0) by simply integrating componentwise:

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s)\, ds$$

(in the above, $+$ denotes simply summation in $\mathbb{R}^{n+1}$) Thus, in order to approximate γ by a polygonal curve γ_ϵ, we will first approximate $\dot{\gamma}$ by a *piecewise constant* curve $\tilde{\gamma}_\epsilon$ and then set

$$\gamma_\epsilon(t) = \gamma(0) + \int_0^t \tilde{\gamma}_\epsilon(s)\, ds.$$

Since γ is C^1, $\dot{\gamma}$ is continuous on $[0,1]$; since, in addition, $[0,1]$ is a compact interval, $\dot{\gamma}$ must also be *uniformly* continuous. Therefore, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any $t, s \in [0,1]$ with $|t-s| < \delta$ we have

$$\sum_{\alpha=0}^{n} \left| \dot{\gamma}^\alpha(t) - \dot{\gamma}^\alpha(s) \right| < \epsilon. \tag{1}$$

For any given $\epsilon > 0$ and $\delta = \delta(\epsilon)$ as above, let us consider the partition $\{t_k\}_{k=0}^N$ of $[0,1]$ with $N = \delta^{-1}$ such that

$$t_k = k\delta.$$

If we define, for $k = 0, \dots, N-1$, the constant vectors

$$\dot{\gamma}_k \doteq \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \dot{\gamma}(t)\, dt,$$

the bound (1) implies that

$$\sup_{t \in [t_k, t_{k+1}]} \sum_{\alpha=0}^{n} \left| \dot{\gamma}^\alpha(t) - \dot{\gamma}_k^\alpha \right| < \epsilon. \tag{2}$$

Note also that, since $\dot{\gamma}(t)$ is a causal, future directed vector and the future timecone is a convex set, the constants $\dot{\gamma}_k$ are also causal and future directed.

Let $v = (1, 0, \dots, 0)$. We will define the piecewise constant curve $\tilde{\gamma} : [0,1) \to \mathbb{R}^{n+1}$ as follows:

$$\tilde{\gamma}_\epsilon(t) = \dot{\gamma}_k(t) + \epsilon v \quad \text{if } t \in [t_k, t_{k+1})$$

and we will set

$$\gamma_\epsilon(t) = \gamma(0) + \int_0^t \tilde{\gamma}_\epsilon(s)\, ds$$

and

$$q_\epsilon = \gamma_\epsilon(1).$$

Note the following facts:

- γ_ϵ is a polygonal curve connecting p to q_ϵ; the polygonal vertices of γ_ϵ are the points $p_k = \gamma_\epsilon(t_k)$ which can be explicitly computed from the formula of γ_ϵ and the fact that $\tilde{\gamma}_\epsilon$ is piecewise constant:
$$p_k = p + \sum_{l=0}^{k-1} \delta \cdot (\dot{\gamma}_l + \epsilon v).$$
- Since $\dot{\gamma}_k$ is future directed causal and v is future directed timelike, $\overrightarrow{p_k p_{k+1}} = \delta \cdot \dot{(\gamma_k} + \epsilon v)$ is future directed and timelike. Therefore, $\overrightarrow{pq_\epsilon} = \sum_{k=0}^{N-1} \overrightarrow{p_k p_{k+1}}$ is also future directed and timelike (thus, $q_\epsilon \in I^+(p)$).
- The definition of $\dot{\gamma}_k$ implies that
$$\int_{t_k}^{t_{k+1}} \dot{\gamma}_k \, dt = \int_{t_k}^{t_{k+1}} \dot{\gamma}(t) \, dt.$$
Therefore, we have
$$\begin{aligned} q_\epsilon &= \gamma_{\epsilon(1)} \\ &= \gamma(0) + \int_0^1 \tilde{\gamma}_\epsilon(s) \, ds \\ &= \gamma(0) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (\dot{\gamma}_k + \epsilon v) \, ds \\ &= \gamma(0) + \epsilon v + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \dot{\gamma}_k \, ds \\ &= \gamma(0) + \epsilon v + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \dot{\gamma}(s) \, ds \\ &= \gamma(0) + \epsilon v + \int_0^1 \dot{\gamma}(s) \, ds \\ &= \gamma(1) + \epsilon v \\ &= q + \epsilon v \end{aligned}$$
and, thus, $q_\epsilon \to q$ as $\epsilon \to 0$.
- For any $k = 0, \ldots, N-1$ and any $t \in (t_k, t_{k+1})$, we have
$$\dot{\gamma}_\epsilon(t) = \dot{\gamma}_k + \epsilon v.$$
Thus, in view of the bound (2) on the difference between $\dot{\gamma}(t)$ and $\dot{\gamma}_k$, we can estimate for any $k = 0, \ldots, N_1$ and any $t \in (t_k, t_{k+1})$
$$\lim_{\epsilon \to 0} \Big(\max_{\substack{k=0,\ldots,N-1 \\ \alpha=0,\ldots n}} \sup_{t \in (t_k, t_{k+1})} \left| \dot{\gamma}^\alpha(t) - \dot{\gamma}_k^\alpha \right| + \epsilon \sum_{\alpha=0}^{n} |v^\alpha| \Big) = 0$$

and, therefore (since $\|\dot{\gamma}\|_\eta$ is a continuous function of $\{\dot{\gamma}^\alpha\}_{\alpha=0}^n$):

$$\lim_{\epsilon\to 0}\Big(\sup_{t\in(0,1)}\Big|\|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta\Big|\Big) \to 0.$$

Therefore:

$$\begin{aligned}\big|\ell(\gamma)-\ell(\gamma_\epsilon)\big| &= \Big|\sum_{k=0}^{N-1}\Big(\int_{t_k}^{t_{k+1}}\big(\|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta\big)\,dt\Big)\Big| \\ &\leqslant \sum_{k=0}^{N-1}(t_{k+1}-t_k)\sup_{t\in(t_k,t_{k+1})}\Big|\|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta\Big| \\ &\leqslant \sup_{t\in(0,1)}\Big|\|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta\Big| \\ &\xrightarrow{\epsilon\to 0} 0.\end{aligned}$$

The last two points above are precisely Conditions 1 and 2 on q_ϵ and γ_ϵ.

2.3 Let (M, g) be a smooth Lorentzian *surface* (i.e. 2-dimensional manifold).

(a) Show that for any $p\in\mathcal{M}$, there exists an open neighborhood $\mathcal{U}$ of p and a local system of coordinates (u, v) on $\mathcal{U}$ such that

$$g = \Omega(u,v)dudv,$$

where $\Omega\in C^\infty(\mathcal{U})$ does not vanish in $\mathcal{U}$ (such a coordinate system is called a *characteristic* or *double null* system).

(b) Deduce that every smooth Lorentzian surface is locally conformally equivalent to an open subset of the Minkowski space $(\mathbb{R}^{1+1},\eta)$ (recall that a similar fact also holds for *Riemannian* surfaces; in that case, a coordinate system exhibiting this equivalence is called *isothermal*).

Solution. (a) For any $p\in\mathcal{M}$, let (x^1,x^2) be a local coordinate system in a neighborhood $\mathcal{V}$ of p; without loss of generality, we can assume that $(x^1(p),x^2(p))=(0,0)$. Since $\mathcal{M}$ is two dimensional, the null cone $N_q\subset T_q\mathcal{M}$ for every $q\in\mathcal{V}$ consists of two intersecting straight lines $\ell_q^{(1)},\ell_q^{2)}\subset T_q\mathcal{M}$. Let us, therefore, choose two smooth *null* vector fields E_1 and E_2 on a (possibly smaller) open neighborhood $\mathcal{V}'$ of p such that $E_i|_q$ spans the null line $l_q^{(i)}$ for every $q\in\mathcal{V}'$. Those vector fields can be constructed explicitly: If, for $i=1,2$, $E_i=E_i^1\frac{\partial}{\partial x^1}+E_i^2\frac{\partial}{\partial x^1}$ are the components of E_i in the (x^1,x^2) coordinate system, then the two vectors $\big(E_i^1(q),E_i^2(q)\big)\in\mathbb{R}^2$ are roots of the quadratic form

$$Q_q(y^1,y^2)=g_{11}(q)(y^1)^2+2g_{12}(q)y^1y^2+g_{22}(y^2)^2$$

on $\mathbb{R}^2$. This quadratic form has coefficients depending smoothly on q (since g was assumed to be smooth) and is of hyperbolic signature since g is Lorentzian (i.e. $g_{12}^2-g_{11}g_{22}<0$); therefore, for any $q \in \mathcal{V}$, $Q_q(y^1, y^2)$ the roots of $Q_q(y^1, y^2)$ consist of two distinct straight lines of $\mathbb{R}^2$ passing through 0 and depending smoothly on q. We can then choose $\big(E_1^1(q), E_1^2(q)\big)$ and $\big(E_2^1(q), E_2^2(q)\big)$ to be generators of these lines satisfying $(E_i^1)^2(q) + (E_i^2)^2(q) = 1$.[1]

We will construct the coordinate functions $u, v : \mathcal{V}'' \subset \mathcal{V} \to \mathbb{R}$ by the requirement that their level sets in $\mathcal{V}''$ (i.e. the curves $\{u = \text{const}\}$ and $\{v = \text{const}\}$) are *integral curves* of the vector fields E_1, E_2, respectively; note that this requirement does not uniquely determine u, v, since any reparametrization of the form $u' = f_1 \circ u$ and $v' = f_2 \circ v$ for smooth and invertible functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ will have the same level sets.

To this end, let $\gamma : \mathbb{R} \to \mathcal{M}$ be a smooth curve such that $\gamma(0) = p$ and $\dot\gamma(0) \in T_p\mathcal{M} \setminus 0$ is transversal to both $E_1|_p, E_2|_p$. By continuity, there exists a $\delta > 0$ such that $\dot\gamma(s)$ is transversal to $E_1|_{\gamma(s)}, E_2|_{\gamma(s)}$ for all $s \in (-\delta, \delta)$. Let us define the following open neighborhoods of p:

$$\mathcal{V}_1 = \big\{ q \in \mathcal{V} : \, q \text{ belongs to an integral curve of } E_1 \text{ passing through } \gamma\big((-\delta,\delta)\big)\big\}$$

and

$$\mathcal{V}_2 = \big\{ q \in \mathcal{V} : \, q \text{ belongs to an integral curve of } E_2 \text{ passing through } \gamma\big((-\delta,\delta)\big)\big\}$$

Let us also set $\mathcal{V}'' = \mathcal{V}_1 \cap \mathcal{V}_2$. Then, we can construct the functions $u, v : \mathcal{V}'' \to \mathbb{R}$ by solving the following initial value problems with initial data on the curve $\gamma\big((-\delta,\delta)\big)$:

$$\begin{cases} E_1(u) = 0, \\ u(\gamma(s)) = s \ \text{ for } s \in (-\delta,\delta) \end{cases} \qquad \begin{cases} E_2(v) = 0, \\ v(\gamma(s)) = s \ \text{ for } s \in (-\delta,\delta). \end{cases}$$

Notice that $(u(p), v(p)) = (0,0)$ and $u = v$ on the curve $\gamma \cap \mathcal{V}'' = \gamma\big((-\delta,\delta)\big)$.

In order to say that (u, v) form a local system of coordinates around p, we have to show that the map $(u, v) : \mathcal{V}'' \to \mathbb{R}^2$ is a diffeomorphism on its image in a neighborhood of p, or, equivalently, that the change of coordinates $(x^1, x^2) \to (u, v)$ is non-singular in a neighborhood of $(x^1(p), x^2(p)) = (0,0)$. By the inverse function theorem, it suffices to show that the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix}$$

is non-degenerate at p. To this end, we have to compute $\partial_i u(p)$ and $\partial_i v(p)$. In view of the definition of the function u, at $p = \gamma(0)$ we have

$$E_1(u)|_p = 0 \quad \text{and} \quad \dot\gamma(0)(u) = 1$$

which can be reexpressed in the (x^1, x^2) coordinate system as

$$\begin{cases} E_1^1 \frac{\partial u}{\partial x_1}(p) + E_1^2 \frac{\partial u}{\partial x_2}(p) = 0 \\ \dot\gamma^1(0) \frac{\partial u}{\partial x_1}(p) + \dot\gamma^2(0) \frac{\partial u}{\partial x_2}(p) = 1 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} E_1^1(p) & E_1^2(p) \\ \dot\gamma^1(0) & \dot\gamma^2(0) \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1}(p) \\ \frac{\partial u}{\partial x_2}(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

[1] A different way of choosing $\big(E_i^1(q), E_i^2(q)\big)$ could be as follows: Assuming without loss of generality that $g_{11}(p) \neq 0$ (and hence $g_{11} \neq 0$ in a neighborhood of p), then, setting $\lambda_i = \frac{E_i^1}{E_i^2}$, we obtain that λ_i must satisfy the quadratic equation $g_{11}(q)\lambda_i^2 + 2g_{12}(q)\lambda_i + g_{22} = 0$, so we can pick $\lambda_1(q)$ and $\lambda_2(q)$ to be the two roots of this equation.

Since we assumed that the vectors $\dot{\gamma}(0), E_1 \in T_p\mathcal{M}$ are transversal (i.e. not collinear), the matrix on the left hand side above has rank 2 and is therefore invertible; we can thus write:

$$\begin{bmatrix} \frac{\partial u}{\partial x_1}(p) \\ \frac{\partial u}{\partial x_2}(p) \end{bmatrix} = \begin{bmatrix} E_1^1(p) & E_1^2(p) \\ \dot{\gamma}^1(0) & \dot{\gamma}^2(0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Working similarly for the function v, we infer:

$$\begin{bmatrix} \frac{\partial v}{\partial x_1}(p) \\ \frac{\partial v}{\partial x_2}(p) \end{bmatrix} = \begin{bmatrix} E_2^1(p) & E_2^2(p) \\ \dot{\gamma}^1(0) & \dot{\gamma}^2(0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the vectors $E_1|_p, E_2|_p$ are not collinear, we can readily compute that the vectors $\begin{bmatrix} \frac{\partial u}{\partial x_1}(p) \\ \frac{\partial u}{\partial x_2}(p) \end{bmatrix}$ and $\begin{bmatrix} \frac{\partial v}{\partial x_1}(p) \\ \frac{\partial v}{\partial x_2}(p) \end{bmatrix}$ obtained from the above expressions (using the formula for the inverse of a 2×2 matrix) are not collinear either; thus, the Jacobian matrix $J(p)$ has rank 2 and is, therefore, invertible. Hence, (u, v) define a coordinate chart in a neighborhood of p.

In the (u, v) coordinate system, the coordinate vector field $\frac{\partial}{\partial u}$ is tangent to the coordinate curve $\{v = const\}$. In view of the fact that $E_2(v) = 0$, this means that $\frac{\partial}{\partial u} \parallel E_2$ and, therefore, $\frac{\partial}{\partial u}$ is *null*, i.e.

$$g_{uu} = g\Big(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\Big) = 0.$$

Similarly, $\frac{\partial}{\partial v} \parallel E_1$ and, therefore,

$$g_{vv} = g\Big(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\Big) = 0.$$

As a result,

$$\begin{aligned} g &= g_{uu}(du)^2 + 2g_{uv}dudv + g_{vv}(dv)^2 \\ &= 2g_{uv}dudv. \end{aligned}$$

Of course, g_{uv} cannot vanish at any point in the region where (u, v) is a coordinate system (if $g_{uv}(q) = 0$, then $g|_q = 0$, which would violate the assumption that $g|_q$ is non-degenerate). Therefore, g_{uv} has a constant sign; by switching $u \to -u$ if necessary, we can assume that $g_{uv} > 0$. Thus, setting

$$\Omega \doteq 2g_{uv},$$

we obtain the required expression.

(b) On Minkowski spacetime $(\mathbb{R}^{1+1}, \eta)$, we can introduce the standard double null coordinate system by setting

$$\begin{aligned} \bar{v} &= x^0 + x^1, \\ \bar{u} &= x^0 - x^1. \end{aligned}$$

In these coordinates, the Minkowski metric η takes the form

$$\begin{aligned}\eta &= -(dx^0)^2 + (dx^1)^2\\ &= -\Big(\frac{d\bar{v}+d\bar{u}}{2}\Big)^2 + \Big(d\frac{\bar{v}-d\bar{u}}{2}\Big)^2\\ &= -d\bar{u}d\bar{v}.\end{aligned}$$

Returning to our 2-dimensional Lorentzian manifold $(\mathcal{M}, g)$, for any point $p \in \mathcal{M}$, let (u, v) be the double null coordinate system in a neighborhood $\mathcal{V}$ of p constructed in part (a); recall that, in these coordinates

$$g = \Omega du dv.$$

Let us define the map $\phi : \mathcal{V} \to \mathbb{R}^{1+1}$ so that, in the $(\bar{u}, \bar{v})$ coordinates on $\mathbb{R}^{1+1}$:

$$\big(\phi^{\bar{u}}(u, v), \phi^{\bar{v}}(u, v)\big) = (u, v))$$

Then we can immediately compute that

$$\begin{aligned}\phi_*\eta \quad &= \quad (\phi_* d\bar{u})(\phi_* d\bar{v})\\ &\overset{\substack{\bar{u}(u,v)=u\\ \bar{v}(u,v)=v}}{=} \quad du dv\end{aligned}$$

and, therefore,

$$\phi_*\eta = \Omega^{-1} \cdot g.$$

Therefore, the map $\phi : (\mathcal{V}, g) \to (\phi(\mathcal{V}), \eta)$ is conformal.

***2.4** In this exercise, we will show that there are topological obstructions to a manifold admitting a Lorentzian metric; **not** every smooth manifold admits one. To this end, let us adopt the following definition: For any Lorentzian inner product space (V, m), we will call any 2-element set of the form $\{u, -u\}$ (where $u \in V \setminus 0$) a *line seed*. A line seed $X = \{u, -u\}$ will be called *causal* if $u \in V$ is a causal vector. We will also define the *trivial* line seed to be the pair $\{0, -0\}$.

Given two causal line seeds $X + \{u, -u\}$ and $Y = \{v, -v\}$, then exactly one of the vectors $+v$ and $-v$ belongs to the same timecone as u. We will define the sum $X + Y$ as the seed $\{u + v, -u - v\}$ if u, v belong to the same time cone and as $\{u - v, -u + v\}$ otherwise. We will extend this definition to include the trivial line seed.

(a) Verify that, with the addition operator defined above, $X_1 + X_2$ is a causal line seed if X_1, X_2 are causal line seeds or if one of them is causal and the other is the trivial line seed.

(b) Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $p \in \mathcal{M}$. Show that there exists an open neighborhood $\mathcal{U}$ of p and a smooth causal vector field $U \in \Gamma(\mathcal{U})$.

(c) A smooth *line field seed* on $\mathcal{M}$ will be an assignment of a line seed $X_p = \{U_p, -U_p\}$ in $T_p\mathcal{M}$ for each $p \in \mathcal{M}$ such that, for any $q \in \mathcal{M}$, there exists an open neighborhood $\mathcal{V}$ of

q and a smooth vector field $Y \in \Gamma(\mathcal{V})$ such that $Y(p) \in X_p$ for all $p \in \mathcal{V}$.[2] Show that $\mathcal{M}$ as above admits a smooth *causal* line field seed.
*Hint: For this part, it might be helpful to use the fact that any smooth manifold admits a **partition of unity**: For any open covering $\{\mathcal{U}_a\}_a$ of $\mathcal{M}$, there exists a family $\{\chi_\beta\}_\beta$ of smooth functions $\chi_\beta : \mathcal{M} \to [0, +\infty)$ satisfying the following properties:*

* *Each χ_β is compactly supported, and its support is contained in one of the open sets $\mathcal{U}_a$.*
* *For each χ_β, supp(χ_β) intersects only finitely many of the supports of χ_γ, $\gamma \neq \beta$.*
* *For any $p \in \mathcal{M}$, $\sum_\beta \chi_\beta(p) = 1$.*

You can then use part 2.4.b to construct a smooth causal line seed field in a neighborhood of every point in $\mathcal{M}$, and then use an appropriate partition of unity to "glue" these constructions together, utilising the notion of the sum of two causal line seeds from part 2.4.a.

(d) Deduce that the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ admits a smooth line subbundle. Can the sphere $\mathbb{S}^2$ admit a Lorentzian metric?
Hint: Use the fact that, for a compact manifold $\mathcal{M}$, if the tangent bundle admits a line subbundle then the Euler characteristic $\chi(\mathcal{M})$ of $\mathcal{M}$ vanishes.

Solution. (a) It is easy to verify using the fact that each connected component C_+ and C_- of the causal cone $C = \{v \in V \setminus 0 : m(v, v) \leqslant 0\}$ is a convex cone that if u, v are causal vectors, then $u + v$ is also a causal vector in the same cone. From this, it readily follows that $X + Y$, as defined in the statement of the exercise, is a causal line seed if X and Y are causal line seeds or if one of them is causal and the other the trivial line seed.

For the rest of this exercise, we will denote with $\mathcal{F}(V)$ the set of line seeds in V which are either causal or trivial. Note that $+$ is well defined on $\mathcal{F}(V) \times \mathcal{F}(V)$ and is associative, commutative and has a unique zero element (the trivial line seed).

(b) Let $(x^0, \dots, x^n)$ be a local system of coordinates in a neighborhood $\mathcal{V}$ of p. Let $U_p \in T_p\mathcal{M} \setminus 0$ be a timelike vector, with corresponding components $\{U_p^\alpha\}_{\alpha=0}^n$. We can then define the smooth vector field U on $\mathcal{V}$ by the relation

$$U = U_p^\alpha \frac{\partial}{\partial x^\alpha},$$

i.e. the components of U in the $(x^0, \dots, x^n)$ coordinate system are constant functions ($U^\alpha = U_p^\alpha$). Since the metric g is smooth, the set of timelike vectors, i.e. the set

$$\mathcal{I} = \{(q, v) \in T\mathcal{M} : g_q(v, v) < 0\}$$

is an open subset of the tangent bundle $T\mathcal{M}$. Therefore, since $U|_p$ was assumed to be timelike vector in $T_p\mathcal{M}$, $U|_q$ will also be timelike (and hence causal) for any point q in a sufficiently small open neighborhood $\mathcal{U}$ of p.

[2]Note that, with this definition, the tangent vector U_p need not even be continuous in p; we only require that there is (locally at least) a choice between U_p and $-U_p$ at every point p that results in a smooth vector field.

(c) From part (b), we know that, for every point $p \in \mathcal{M}$, there exists an open neighborhood $\mathcal{U}^{(p)}$ of p and a smooth causal vector field $T^{(p)}$ defined on $\mathcal{U}^{(p)}$. We can, therefore, also define the causal line field seed $X^{(p)}$ on $\mathcal{U}^{(p)}$ by the relation

$$X^{(p)}|_q = \{T^{(p)}|_q, -T^{(p)}|_q\} \text{ for any } q \in \mathcal{U}^{(p)}.$$

The collection of open sets $\{\mathcal{U}^{(p)}\}_{p\in\mathcal{M}}$ covers the whole of $\mathcal{M}$ (since, for any $q \in \mathcal{M}$, $q \in \mathcal{U}^{(q)}$). We can therefore introduce a partition of unity $\{\chi_\beta\}_\beta$ subordinate to the open cover $\{\mathcal{U}^{(p)}\}_{p\in\mathcal{M}}$; this is a set of smooth functions $\chi_\beta : \mathcal{M} \to [0, +\infty)$ (where the set of indices β is not necessarily the same as the index set for the open cover) satisfying the following properties:

1. For any β, there exists a $p = p(\beta)$ such that $\text{supp}(\chi_\beta) \subset U^{(p)}$.

2. For any $p \in \mathcal{M}$, there exists an open neighborhood $\mathcal{V}_p$ of p such that only finitely many of the functions χ_β are supported on $\mathcal{V}_p$.

3. For any $p \in \mathcal{M}$, we have
$$\sum_\beta \chi_\beta(p) = 1$$
(by the previous property, this is a finite sum).

The fact that such a partition of unity always exists for any open cover of a smooth manifold is a fundamental result in the theory of manifolds; see for example the book by Brickel and Clark: *Differentiable manifolds: An Introduction.*

For a partition of unity $\{\chi_\beta\}_\beta$ as above, let us define a (non-unique) map $\beta \to p(\beta) \in \mathcal{M}$ so that $\text{supp}\chi_\beta \subset \mathcal{U}^{(p)}$ (such a p exists by property 1 above). For any β, let us consider the pair of vector fields defined on $\mathcal{U}^{(p(\beta))}$

$$X_\beta = \{\chi_\beta \cdot T^{(p(\beta))}, -\chi_\beta \cdot T^{(p(\beta))}\}.$$

Since the support of χ_β is contained in $\mathcal{U}^{(p(\beta))}$, the vector fields $\pm\chi_\beta \cdot T^{(p(\beta))}$ can be smoothly extended on the whole of $\mathcal{M}$ by simply assuming that they vanish identically on $\mathcal{M} \setminus \mathcal{U}^{(p(\beta))}$. In this way, the pair X_β is now a pair of smooth vector fields on the whole of $\mathcal{M}$; at any point $q \in \mathcal{M}$, $X_\beta|_q \in \mathcal{F}(T_q\mathcal{M})$ (see the end of the solution of part (a) for the notation $\mathcal{F}(V)$) and, moreover, at any point q such that $\chi_\beta(q) > 0$, the pair $X_\beta|_q$ is a *causal line seed* (note that X_β is the trivial line seed on $\overline{\mathcal{M} \setminus \text{supp}\chi_\beta}$).

Let us consider, now, for any point $q \in \mathcal{M}$, the following element of $\mathcal{F}(T_q\mathcal{M})$ (see, again, the end of the solution of part (a) for the and the addition operation on $\mathcal{F}(V)$):

$$X|_q = \sum_\beta X_\beta|_q.$$

This is a finite sum, since only a finite number of the X_β's are non-zero at q; moreover, the sum is well defined as an element of $\mathcal{F}(T_q\mathcal{M})$ because, for all β, $X_\beta|_q \in \mathcal{F}(T_q\mathcal{M})$. Moreover, the following properties hold:

- For any point $p \in \mathcal{M}$, using the timelike vector field $T^{(p)}$ defined on $\mathcal{U}^{(p)}$, we can define a continuous assignment of a future directed causal cone to the tangent spaces $T_q\mathcal{M}$ for all $q \in \mathcal{U}^{(p)}$: For any $q \in \mathcal{U}^{(p)}$ the *future directed* component C_q^+ of the causal cone $C_q = \{v \in T_q\mathcal{M} : v \text{ is } \textit{causal}\}$ can be fixed by the condition:

$$C_q^+ = \big\{v \in C_q : \, g(v, T^{(p)}) < 0).$$

(since $T^{(p)}$ is a smooth vector field, this assignment of a future directed component is continuous in $q \in \mathcal{W}_p$). Therefore, for any one of the vector field pairs $X_\beta = \big\{\chi_\beta \cdot T^{(p(\beta))}, -\chi_\beta \cdot T^{(p(\beta))}\big\}$ restricted over supp$\chi_{\beta(p)}$, we can distinguish one future directed and one past directed element of $X_\beta|_{\mathcal{U}^{(p)}}$; we will denote with $X_\beta|^+_{\mathcal{U}^{(p)}}$ the future directed vector field among the pair $\big\{\chi_\beta \cdot T^{(p(\beta))}|_{\mathcal{U}^{(p)}}, -\chi_\beta \cdot T^{(p(\beta))}|_{\mathcal{U}^{(p)}}\big\}$ and with $X_\beta|^-_{\mathcal{U}^{(p)}}$ the past directed one. We will extend this definition outside supp$\chi_{\beta(p)}$ trivially, since $X_\beta = 0$ there. Thus,

$$X_\beta|_{\mathcal{U}^{(p)}} = \Big\{X_\beta|^+_{\mathcal{U}^{(p)}}, \, X_\beta|^-_{\mathcal{U}^{(p)}}\Big\}.$$

Having this distinction between future and past directed causal vector fields over $\mathcal{U}^{(p)}$, it is then easy to see that X restricted to $\mathcal{U}^{(p)}$ can be written as

$$\begin{aligned} X|_{\mathcal{U}^{(p)}} &= \Big\{X|^+_{\mathcal{U}^{(p)}}, \, X|^-_{\mathcal{U}^{(p)}}\Big\} \\ &= \Big\{\sum_\beta X_\beta|^+_{\mathcal{U}^{(p)}}, \, \sum_\beta X_\beta X|^-_{\mathcal{U}^{(p)}}\Big\}. \end{aligned}$$

In particular, the line seed $X|_{\mathcal{U}^{(p)}}$ can be written as a pair of smooth vector fields; therefore X is a *smooth* line field.

- For any point $p \in \mathcal{M}$, there exists at least one β such that $\chi_\beta(p) > 0$ (since $\sum_\beta \chi_\beta(p) = 1$). Therefore, not all vector fields $X_\beta|^+_{\mathcal{U}(p)}$ vanish at p; as a result, $X|_p \in \mathcal{F}(T_p\mathcal{M})$ is not the pair $\{0, 0\}$ and is therefore a *causal* line seed.

Thus, we have shown that X is a *smooth*, *causal* line seed field on $\mathcal{M}$.

(d) The existence of a smooth line seed field X on $\mathcal{M}$ determines a line subbundle of $T\mathcal{M}$, namely the subbundle $E \hookrightarrow T\mathcal{M}$ spanned by the two elements of $X|_p = \{x_p, -x_p\}$ at each point $p \in \mathcal{M}$:

$$E = \{(p, v) \in T\mathcal{M} : \, v = \lambda x_p \text{ for some } \lambda \in \mathbb{R} \text{ and } x_p \in X|_p\}.$$

It is known from algebraic topology that, if the tangent bundle $T\mathcal{M}$ of a smooth compact manifold $\mathcal{M}$ admits a smooth line subbundle, then the *Euler characteristic* of the manifold $\mathcal{M}$ (which can be computed as the alternating sum $\sum_{k=0}^n (-1)^k F_k$ of the k-dimensional faces in a finite triangulation of the manifold $\mathcal{M}$). The Euler characteristic of the sphere $\mathbb{S}^n$ is $1 + (-1)^n$; thus, for $n \in 2\mathbb{Z}$, the tangent bundle of the sphere $\mathbb{S}^n$ cannot admit a smooth line subbundle and, therefore, $\mathbb{S}^n$ in this case cannot admit a smooth Lorentzian metric.

Bonus exercise (hard): Can you construct a Lorentzian metric on $\mathbb{S}^3$? (*Hint: Use the Hopf fibration* $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ to foliate $\mathbb{S}^3$ by 1-dimensional circles and use a vector field tangent to those circles to construct suitable timecones.)